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THE SPATIAL PROBLEM OF THE COMPRESSION OF A MATERIAL ALONG A PERIODIC SYSTEM OF PARALLEL CIRCULAR CRACKS*

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The non-axisymmetric problem of the biaxial uniform compression of a material along a periodic system of parallel circular cracks is considered. A facture criterion is used /1, 2/ within the framework of linearized stability theory according to which the beginning of fracture of the material under compression along the cracks is characterized by local buckling near the cracks. Within the framework of this approach, axisymmetric and plane problems were considered earlier for different material models (highly-elastic, composite and plastic) for one or two internal cracks, near-surface cracks and a periodic system of cracks /1-13/**. (**See also: Nazarenko, V.M., The axisymmetric problem of the fracture mechanics of materials under compression along a periodic system of parallel cracks (unequal roots).

Proceeding of the Eleventh Scientic Conf. of Young Scientists. Inst. Mechanics, Ukraine Academy of Sciences, Kiev, 1986. 154-161, Dep. VINITI 5507-86, July 28, 1986 Nazarenko, V.M. and Starodubtsev, I.P., On material fracture under compression along two parallel cracks in the case of plane strain. Non-classical and Mixed Problems of the Mechanics of a Deformable Body: Materials of a Seminar of Young Scientists, Kiev, 1985, 142-145, Dep. 5531-85 in VINITI, July 29, 1985.) The investigation is performed in general form for an arbitrary kind of elastic potential for compressible and incompressible materials, the theory of large and modifications of small subcritical strains, and can be extended to other models of a deformable body (composites, plastic bodies, etc.).

1. Formulation of the problem. Fracture of a material weakened by a periodic system of parallel disc-shaped coaxial cracks $\{r < a, 0 \leqslant \theta < 2\pi, x_3 = 2hn, n = 0, \pm 1, \pm 2, \ldots\}$ under biaxial compression in planes parallel to the cracks is considered. Lagrange coordinates x_j $(j = 1, 2, 3)$ are utilized that are identical with the Cartesian coordinates in the undeformed state, as are the symmetric stress tensor $\mathbf{S}^0$ referred to unit area of the body in the undeformed state, $\mathbf{u}$, $\mathbf{t}$ is the perturbation of the displacement vector and the non-symmetric Kirchhoff stress tensor, respectively, and r, θ, x_3 are the cylindrical coordinates obtainable from the Cartesian coordinates x_j $(j = 1, 2, 3)$.

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The realizable homogeneous subcritical state is given by the relationships /1/

$$S_{33}^{\circ} = 0,\ S_{11}^{\circ} = S_{22}^{\circ} \neq 0,\ S_{11}^{\circ} = \text{const} \tag{1.1}$$
$$u_j^{\circ} = \delta_{jm} (\lambda_j - 1) x_m;\quad \lambda_j = \text{const};\quad \lambda_1 = \lambda_2 \neq \lambda_3$$

(λ_j are the elongations along the axes $\lambda_1 < 1$). The crack edges are stress-free. The boundary conditions of the linearized problem have the form

$$t_{33} = t_{3r} = t_{3\theta} = 0 (x_3 = (2hn)_{\pm},\ r < a,\ 0 \leqslant \theta < 2\pi) \tag{1.2}$$
$$n = 0,\ \pm 1,\ \pm 2,\ \ldots,$$

where the plus and minus subscripts denote the appropriate crack edges.

In view of the periodicity of the geometric and force schemes of the problem, also considering the symmetric and antisymmetric stress and displacement fields relative to this plane separately (because of the linearity of the problem and its symmetry relative to the plane $x_3 = 0$), we can reduce the initial problem to a problem for the layer $0 \leqslant x_3 \leqslant h$ separately for the symmetric and bending buckling modes with the following boundary conditions $0 \leqslant \theta < 2\pi$ everywhere):

The symmetric mode

$$u_3 = 0\ (x_3 = 0,\ r > a),\ t_{33} = 0\ (x_3 = 0,\ r < a) \tag{1.3}$$
$$t_{3r} = t_{3\theta} = 0\ (x_3 = 0,\ 0 \leqslant r < \infty)$$
$$u_3 = 0,\ t_{3r} = t_{3\theta} = 0\ (x_3 = h,\ 0 \leqslant r < \infty)$$

The bending mode

$$u_r = u_\theta = 0\ (x_3 = 0,\ r > a) \tag{1.4}$$
$$t_{3r} = t_{3\theta} = 0\ (x_3 = 0,\ r < a)$$
$$t_{33} = 0\ (x_3 = 0,\ 0 \leqslant r < \infty)$$
$$u_r = u_\theta = 0,\quad t_{33} = 0\quad (x_3 = h,\quad 0 \leqslant r < \infty)$$

Investigation of problems (1.3) and (1.4) relies upon the apparatus of the theory of cracks for bodies with initial stresses /1/.

Representations of the general solutions of the linearized problem for the initial states (1.1) in terms of potential functions in a circular cylindrical coordinate system are given by the relationships /1, 6/ (limited to the case of equal roots $n_1^{\circ} = n_2^{\circ}$ of the characteristic equation, the terminology and notation in /1, 6/)

$$u_r = -\frac{\partial \varphi}{\partial r} - z_1 \frac{\partial F}{\partial r} - \frac{1}{r}\frac{\partial \varphi_3}{\partial \theta},\quad u_\theta = -\frac{1}{r}\frac{\partial \varphi}{\partial \theta} - z_1 \frac{1}{r}\frac{\partial F}{\partial \theta} + \frac{\partial \varphi_3}{\partial r} \tag{1.5}$$
$$u_3 = (n_1^{\circ})^{-1/2} (m_1^{\circ} - m_2^{\circ} + 1) F - m_1^{\circ} (n_1^{\circ})^{-1/2} \Phi - m_1^{\circ} (n_1^{\circ})^{-1/2} z_1 \frac{\partial F}{\partial z_1}$$
$$t_{33} = c_{44}^{\circ} \left[(d_1 l_1^{\circ} - d_2 l_2^{\circ}) \frac{\partial F}{\partial z_1} - d_1 l_1^{\circ} \frac{\partial \Phi}{\partial z_1} - d_1 l_1^{\circ} z_1 \frac{\partial^2 F}{\partial z_1^2} \right]$$
$$t_{3\theta} = c_{44}^{\circ} \left\{ (n_1^{\circ})^{-1/2} \frac{1}{r}\frac{\partial}{\partial \theta} [(d_1 - d_2) F - d_1 \Phi] - (n_1^{\circ})^{-1/2} d_1 z_1 \frac{1}{r} \frac{\partial^2 F}{\partial \theta \partial z_1} + (n_3^{\circ})^{-1/2} d_3 \frac{\partial^2 \varphi_3}{\partial r \partial z_3} \right\},$$
$$t_{3r} = c_{44}^{\circ} \left\{ (n_1^{\circ})^{-1/2} \frac{\partial}{\partial r} [(d_1 - d_2) F - d_1 \Phi] - (n_1^{\circ})^{-1/2} d_1 z_1 \frac{\partial^2 F}{\partial r \partial z_1} - (n_3^{\circ})^{-1/2} d_3 \frac{1}{r} \frac{\partial^2 \varphi_3}{\partial \theta \partial z_3} \right\}$$

The potential functions $\varphi (r, \theta, z_1)$, $\Phi (r, \theta, z_1)$, $F (r, \theta, z_1)$, $\varphi_3 (r, \theta, z_3)$ are harmonic functions of their arguments and

$$\Phi \equiv \partial \varphi / \partial z_1;\ z_i = (n_i^{\circ})^{-1/2} x_3,\ i = 1, 3 \tag{1.6}$$

The quantities n_1°, n_3°, m_j°, l_j°, d_j ($j = 1, 2$) in the representations (1.5) are determined by selecting the elastic potential (with appropriate simplifications for modifications of the theory of small subcritical strains) or in the general case by the selection of the deformable body model /1/. We proceed as is customary in classical elasticity theory when investigating non-axisymmetric problems /14-16/; we represent the harmonic potential functions in the form of series in the harmonics of argument θ with coefficients in the form of Hankel integral expansions in the coordinate r of order corresponding to the number of the harmonic (here and henceforth summation is over n from 0 to ∞)

$$\varphi(r,\theta,z_1) = -\sum \cos n\theta \int_0^\infty [B_n^{(1)}(\lambda)\operatorname{sh}\lambda(h_1 - z_1) + B_n^{(2)}(\lambda)\operatorname{ch}\lambda(h_1 - z_1)]\, J_n(\lambda r)\frac{d\lambda}{\lambda \operatorname{sh}\lambda h_1} \tag{1.7}$$

$$F(r,\theta,z_1) = \sum \cos n\theta \int_0^\infty [A_n^{(1)}(\lambda)\operatorname{ch}\lambda(h_1 - z_1) + A_n^{(2)}(\lambda)\operatorname{sh}\lambda(h_1 - z_1)]\, J_n(\lambda r)\frac{d\lambda}{\operatorname{sh}\lambda h_1}$$

$$\varphi_3(r,\theta,z_3) = \sum \sin n\theta \int_0^\infty [C_n^{(1)}(\lambda)\operatorname{ch}\lambda(h_3 - z_3) + C_n^{(2)}(\lambda)\operatorname{sh}\lambda(h_3 - z_3)]\, J_n(\lambda r)\frac{d\lambda}{\lambda \operatorname{sh}\lambda h_3}$$

$$(h_j = (n_j^\circ)^{-1/2} h,\ j = 1, 3)$$

where $A_n^{(i)}, B_n^{(i)}, C_n^{(i)}$ are unknown functions. The representation for the function $\Phi(r,\theta,z_1)$ follows from (1.6)

2. System of dual integral equations. We will investigate problems (1.3) and (1.4) for the symmetric and bending buckling modes by first reducing them to dual integral equations for each harmonic, and then to Fredholm integral equations of the second kind. In view of the awkwardness of the calculations, we will represent the investigation procedure by the example of problem (1.4) for the bending mode.

Satisfying the boundary conditions given in the whole plane $x_3 = \text{const}$ (or $z_i = \text{const}$, $i = 1, 3$) - the last two rows in (1.4), we obtain four equations connecting the functions $A_n^{(j)}, B_n^{(j)}, C_n^{(j)}$, $j = 1, 2$. Utilizing the relationship

$$J_\nu'(z) = -J_{\nu+1}(z) + \nu z^{-1} J_\nu(z) \tag{2.1}$$

and equating the expressions for $\sin n\theta$ and $\cos n\theta$ to zero, we obtain that the equations mentioned are satisfied identically if

$$C_n^{(1)} = 0,\quad B_n^{(2)} = \lambda h_1 A_n^{(1)},\quad A_n^{(2)} = 0,$$
$$B_n^{(1)} = \left[\left(1 - \frac{d_2 l_2^\circ}{d_1 l_1^\circ}\right) - \lambda h_1 \operatorname{cth}\lambda h_1\right] A_n^{(1)} \tag{2.2}$$

The remaining boundary conditions in (1.4) (the first two rows) result in a system of dual equations (taking into account (1.5), (1.7) and (2.2))

$$\Sigma \cos n\theta \int_0^\infty \left[(B_n^{(1)} + B_n^{(2)}\operatorname{cth}\lambda h_1)\frac{\partial J_n(\lambda r)}{\partial(\lambda r)} - \frac{n}{\lambda r} C_n^{(2)} J_n(\lambda r)\right] d\lambda = 0,\quad r > a \tag{2.3}$$

$$\Sigma \sin n\theta \int_0^\infty \left[-\frac{n}{\lambda r}(B_n^{(1)} + B_n^{(2)}\operatorname{cth}\lambda h_1) J_n(\lambda r) + C_n^{(2)}\frac{\partial J_n(\lambda r)}{\partial(\lambda r)}\right] d\lambda = 0,\quad r > a$$

$$\Sigma \cos n\theta \int_0^\infty \Big\{(n_1^\circ)^{-1/2}\big[(d_1 - d_2) A_n^{(1)}\operatorname{cth}\lambda h_1 - d_1(B_n^{(1)}\operatorname{cth}\lambda h_1 + B_n^{(2)})\big]\frac{\partial J_n(\lambda r)}{\partial(\lambda r)} + (n_3^\circ)^{-1/2} d_3 \frac{n}{\lambda r} C_n^{(2)}\operatorname{cth}\lambda h_3 J_n(\lambda r)\Big\}\lambda\, d\lambda = 0,\quad r < a$$

$$\Sigma \sin n\theta \int_0^\infty \Big\{-(n_1^\circ)^{-1/2}\frac{n}{\lambda r}\big[(d_1 - d_2) A_n^{(1)}\operatorname{cth}\lambda h_1 - d_1(B_n^{(1)}\operatorname{cth}\lambda h_1 + B_n^{(2)})\big] J_n(\lambda r) - (n_3^\circ)^{-1/2} d_3 C_n^{(2)}\operatorname{cth}\lambda h_3 \frac{\partial J_n(\lambda r)}{\partial(\lambda r)}\Big\}\lambda\, d\lambda = 0,\quad r < a$$

We note that on selecting the general solutions in the form of (1.5) when the potential functions are chosen in the form (1.7) of a sum of trigonometric functions multiplied by the Hankel integral expansions, the components $u_r, u_\theta, t_{3r}, t_{3\theta}$ taking the relationships (2.1) and

$$2\nu z^{-1}J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z)$$

into account can be represented in general form (analogous to the representation /14/ in the classical case) as

$$u_r = \Sigma \cos n\theta (K_{n+1} - L_{n-1}), \quad u_\theta = \Sigma \sin n\theta (K_{n+1} + L_{n-1})$$

$$t_{3r} = c_{44}^\circ \Sigma \cos n\theta (U_{n+1} - V_{n-1}), \quad t_{3\theta} = -c_{44}^\circ \Sigma \sin n\theta (U_{n+1} + V_{n-1})$$

Here K_{n+1}, U_{n+1} are Hankel integral transforms of order $(n+1)$, and L_{n-1}, V_{n-1} are of order $(n-1)$. Consequently, two conditions of the type $u_r = 0, u_\theta = 0$ (or $t_{3r} = 0, t_{3\theta} = 0$) for $z_l = \mathrm{const}$ can be reformulated to the form $K_{n+1} = 0, L_{n-1} = 0$ (or $U_{n+1} = 0, V_{n-1} = 0$) for $z_l = \mathrm{const}$ by equating the coefficients of $\sin n\theta$ and $\cos n\theta$ to zero.

Taking account of the above, we conclude that the dual Eqs.(2.3) decompose into separate equations corresponding to each n-th harmonic in the variable θ

$$\int_0^\infty X_\pm J_{n\pm1}(\lambda r)\,d\lambda = 0, \quad r > a; \quad X_\pm = \left(1 - \frac{d_2 l_2^c}{d_1 l_1^\circ}\right) A_n^{(1)} \pm C_n^{(2)} \tag{2.4}$$

$$\int_0^\infty \{(n_1^\circ)^{-1/2} d_1 A_n^{(1)} [-k \operatorname{cth} \mu_1 + \mu_1 (\operatorname{cth}^2 \mu_1 - 1)] \mp \tag{2.5}$$

$$(n_3^\circ)^{-1/2} d_3 C_n^{(2)} \operatorname{cth} \mu_3\} J_{n\pm1}(\lambda r) \lambda\, d\lambda = 0, \quad r < a$$

$n = 1, 2, \ldots; \quad \mu_j = \lambda h_j, \quad j = 1, 3; \quad k = (l_1^\circ - l_2^\circ) d_2 (d_1 l_1^\circ)^{-1}$

The case of the axisymmetric problem $(n = 0)$ for the system of dual equations (2.4) and (2.5) is singular since in this case only equations with the upper signs should be retained in the system by setting $C_0^{(2)} = 0$ therein, since $\sin n\theta \equiv 0$ for $n = 0$, see (2.3). The result also follows directly from system (2.4) and (2.5) for $n = 0$ if it is taken into account that $J_{-1}(z) = -J_1(z)$. The axisymmetric case is discussed in detail earlier /13/ and, consequently, we will not discuss it further.

3. Solution of the system of dual equations. Obtaining Fredholm integral equations. One of the methods of solving dual integral equations is the method of substitution which is this case consists of the fact that $X_\pm$ are selected in a form such that relationships (2.4) are satisfied identically. The two remaining relationships (2.5) are usually converted to Abel integral equations by special methods (the method is demonstrated in /15/ for non-axisymmetric problems) or into Schloemilch integral equations (see /17/, where dual equations are considered for Bessel functions of identical order), whose solutions also yield the desired Fredholm equations of the second kind.

The method described in /17/, based on obtaining the Schloemilch equation modernized to the case of the dual equations (2.4) and (2.5) in which Bessel functions of the different orders $((n+1)$ and $(n-1))$ occur, will be used below.

We will represent the solution of the system of dual integral equations under consideration in the form ($\varphi_\pm(t)$ are unknown functions)

$$X_\pm = \left(\frac{\pi}{2}\right)^{1/2} \lambda^{1\pm1/2} \int_0^a t^{1/2} \varphi_\pm(t) J_{n\pm1/2}(\lambda t)\,dt \tag{3.1}$$

We later need the relationship /18/

$$t^{\mp\nu} \frac{d}{dt}[t^{\pm\nu} J_\nu(\xi t)] = \pm \xi J_{\nu\mp1}(\xi t) \tag{3.2}$$

which is a special case of the discontinuous Webber-Shafheitlin integral and the Sonine integral

$$\int_0^\infty \lambda^{1/2} J_\nu(\lambda x) J_{\nu-1/2}(\lambda y)\,d\lambda = \begin{cases} 0, & 0 < x < y \\ \left(\frac{2}{\pi}\right)^{1/2} \dfrac{y^{\nu-1/2}}{x^\nu (x^2 - y^2)^{1/2}}, & 0 < y < x \end{cases} \tag{3.3}$$

$$\int_0^{\pi/2} \sin^n \theta J_{n-1}(\lambda x \sin\theta)\,d\theta = \left(\frac{\pi}{2\lambda x}\right)^{1/2} J_{n-1/2}(\lambda x) \tag{3.4}$$

Selection of the solution in the form (3.1) enables relationships (2.4) to be satisfied by taking account of (3.2) and (3.3) and integrating by parts. We express $A_n^{(1)}$, and $C_n^{(2)}$ in terms of $X_\pm$. Substitution into (2.5) yields the equation

$$\int_0^\infty (P_\pm X_+ + P_\mp X_-) J_{n\pm1}(\lambda r)\lambda\, d\lambda = 0, \quad r < a \tag{3.5}$$

$$P_\pm = \frac{1}{2}[-ks \mp q + s(-kI(\mu_1) + J(\mu_1)) \mp qI(\mu_3)]$$

$$s = (n_1^\circ)^{-1/2} d_1 \left(1 - \frac{d_3 l_3^\circ}{d_1 l_1^\circ}\right)^{-1}, \quad q = (n_3^\circ)^{-1/2} d_3$$

$$I(\mu) = \frac{e^{-\mu}}{\operatorname{sh}\mu}, \quad J(\mu) = \frac{\mu}{\operatorname{sh}^2\mu}$$

We convert (3.5) first to Schloemilch integral equations and then to Fredholm integral equations of the second kind. We will demonstrate the procedure mentioned in an example of (3.5) with the upper signs.

Using (3.2) and integration by parts, we represent

$$X_+ = -\left(\frac{\pi}{2}\right)^{1/2} \lambda^{1/2} \left\{a^{1/2}\varphi_+(a) J_{n-1/2}(\lambda a) - \int_0^a t^{-n+1/2}\psi_+(t) J_{n-1/2}(\lambda t)\, dt\right\} \tag{3.6}$$

$$\lambda J_{n+1}(\lambda r) = -r^n \frac{d}{dr}[r^{-n} J_n(\lambda r)]$$

and using (3.3) and (3.6) we obtain

$$I_\pm \equiv \int_0^\infty X_\pm \lambda J_{n+1}(\lambda r)\, d\lambda = -r^n \frac{d}{dr} r^{-2n} \int_0^r \psi_\pm(t) \frac{dt}{(r^2 - t^2)^{1/2}} \tag{3.7}$$

$$\psi_+(t) = \frac{d}{dt}[t^n \varphi_+(t)], \quad \psi_-(t) = t^n \varphi_-(t)$$

$$J_\pm \equiv \int_0^\infty L_\pm(\lambda) X_\pm \lambda J_{n+1}(\lambda r)\, d\lambda, \quad L_\pm(\lambda) =$$

$$s[-kI(\mu_1) + J(\mu_1)] \mp qI(\mu_3)$$

Eq.(3.5) with the upper signs which has the form in the notation (3.7)

$$(-ks - q) I_+ + (-ks + q) I_- + J_+ + J_- = 0$$

will take the following form after multiplication by r^{-n}, integration with respect to r between 0 and r and multiplication by r^n:

$$(ks + q) r^{-n} \int_0^r \psi_+(t) \frac{dt}{(r^2 - t^2)^{1/2}} + (ks - q) r^{-n} \int_0^r \psi_-(t) \frac{dt}{(r^2 - t^2)^{1/2}} + \tag{3.8}$$

$$r^n \int_0^r \rho^{-n} \int_0^\infty L_+(\lambda) X_+ \lambda J_{n+1}(\lambda\rho)\, d\lambda\, d\rho +$$

$$r^n \int_0^r \rho^{-n} \int_0^\infty L_-(\lambda) X_- \lambda J_{n+1}(\lambda\rho)\, d\lambda\, d\rho = 0$$

Using the relationship

$$\int_0^r \frac{d}{dr}[r^{-n} J_n(\lambda r)]\, dr = r^{-n} J_n(\lambda r) - \lim_{\rho \to 0} \rho^{-n} J_n(\lambda\rho) =$$

$$r^{-n} J_n(\lambda r) - \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n$$

and substuting $t = r\sin\theta$ we obtain from (3.8)

$$(ks + q) \int_0^{\pi/2} \psi_+(r\sin\theta)\, d\theta + (ks - q) \int_0^{\pi/2} \psi_-(r\sin\theta)\, d\theta = N(r) \tag{3.9}$$

$$N(r) = r^n \int_0^\infty (L_+(\lambda) X_+ + L_-(\lambda) X_-) \left[J_n(\lambda r) - \frac{1}{n!}\left(\frac{\lambda r}{2}\right)^n\right] d\lambda$$

The Schloemilch equation

$$\int_0^{\pi/2} f(r\sin\theta)\, d\theta = N(r) \quad (0 \leqslant r \leqslant a)$$

has the solution

$$f(x) = \frac{2}{\pi}\left[N(0) + x\int_0^{\pi/2} N'(x\sin\theta)\,d\theta\right]$$

In the case under consideration

$$f(x) = (ks+q)\psi_+(x) + (ks-q)\psi_-(x),\quad N(0) = 0$$

$$\int_0^{\pi/2} N'(x\sin\theta)\,d\theta = \int_0^\infty (L_+(\lambda)X_+ + L_-(\lambda)X_-)\times$$
$$\left[x^{n-1/2}\lambda^{1/2}\left(\frac{\pi}{2}\right)^{1/2} J_{n-1/2}(\lambda x) - \frac{2n}{n!}\left(\frac{\lambda}{2}\right)^n x^{2n-1}\int_0^{\pi/2}\sin^{2n-1}\theta\,d\theta\right]d\lambda$$

(using the Sonine integral (3.4)). Taking account of the relationship /19/

$$\int_0^{\pi/2}\sin^{2\nu-1}\theta\,d\theta = \frac{\pi^{1/2}}{2}\frac{\Gamma(\nu)}{\Gamma(\nu+1/2)}$$

and using the representation for X_+ in the form (3.6) and X_- in the form (3.1), we obtain the desired Fredholm equation of the second kind

$$(ks+q)\psi_+(x) + (ks-q)\psi_-(x) + \frac{2}{\pi}\int_0^a \psi_+(t)M_{11}(x,t)\,dt + \qquad (3.10)$$
$$\frac{2}{\pi}\int_0^a \psi_-(t)M_{12}(x,t)\,dt = 0,\quad 0 \leqslant x \leqslant a,\quad n = 1, 2, 3, \ldots$$

with the kernels

$$M_{11}(x,t) = \pi x\int_0^\infty L_+(\lambda)\left(\frac{\lambda}{2}\right)^{1/2}\left[t^{-n+1/2}J_{n-1/2}(\lambda t) - a^{-n+1/2}J_{n-1/2}(\lambda a)\right]\times \qquad (3.11)$$
$$\left[\left(\frac{\lambda}{2}\right)^n\frac{x^{2n-1}}{\Gamma(n+1/2)} - x^{n-1/2}\left(\frac{\lambda}{2}\right)^{1/2}J_{n-1/2}(\lambda x)\right]d\lambda$$
$$M_{12}(x,t) = \pi x t^{-n+1/2}\int_0^\infty L_-(\lambda)\left(\frac{\lambda}{2}\right)^{1/2}J_{n-1/2}(\lambda t)\left[\left(\frac{\lambda}{2}\right)^n\frac{x^{2n-1}}{\Gamma(n+1/2)} - \right.$$
$$\left. x^{n-1/2}\left(\frac{\lambda}{2}\right)^{1/2}J_{n-1/2}(\lambda x)\right]d\lambda$$

In exactly the same way, we obtain the second Fredholm equation

$$(ks-q)\psi_+(x) + (ks+q)\psi_-(x) + \frac{2}{\pi}\int_0^a \psi_+(t)M_{21}(x,t)\,dt + \qquad (3.12)$$
$$\frac{2}{\pi}\int_0^a \psi_-(t)M_{22}(x,t)\,dt = 0,\quad 0 \leqslant x \leqslant a,\quad n = 1, 2, 3, \ldots$$

with the kernels

$$M_{21}(x,t) = \frac{\pi}{2}x^{n+1/2}\int_0^\infty L_-(\lambda)\lambda\left[a^{-n+1/2}J_{n-1/2}(\lambda a) - \right. \qquad (3.13)$$
$$\left. t^{-n+1/2}J_{n-1/2}(\lambda t)\right]J_{n-1/2}(\lambda x)\,d\lambda$$
$$M_{22}(x,t) = -\frac{\pi}{2}x^{n+1/2}t^{-n+1/2}\int_0^\infty L_+(\lambda)\lambda J_{n-1/2}(\lambda t)J_{n-1/2}(\lambda x)\,d\lambda$$

from (3.5) with the lower signs.

Writing the half-sum and the half-difference of (3.10) and (3.12), we obtain the resolving system of integral equations in the form

$$\psi_+(x) + \psi_-(x) + \frac{2}{\pi}\int_0^a \psi_+(t)K_{11}(x,t)\,dt + \frac{2}{\pi}\int_0^a \psi_-(t)K_{12}(x,t)\,dt = 0 \qquad (3.14)$$
$$\psi_+(x) - \psi_-(x) + \frac{2}{\pi}\int_0^a \psi_+(t)K_{21}(x,t)\,dt + \frac{2}{\pi}\int_0^a \psi_-(t)K_{22}(x,t)\,dt = 0$$

$$0 \leqslant x \leqslant a, \ n = 1, 2, 3, \ldots$$
$$K_{11}(x, t) = [M_{11}(x, t) + M_{21}(x, t)]/(2ks), \ K_{12}(x, t) = [M_{12}(x, t) + M_{22}(x, t)]/(2ks)$$
$$K_{21}(x, t) = [M_{11}(x, t) - M_{21}(x, t)]/(2q), \ K_{22}(x, t) = [M_{12}(x, t) - M_{22}(x, t)]/(2q)$$

Investigation of problem (1.3) for the symmetric mode results /17/ in the equation

$$\varphi(x) - \frac{2}{\pi}\int_0^a \varphi(t) K(x, t)\, dt = 0, \quad 0 \leqslant x \leqslant a, \quad n = 1, 2, 3 \ldots \tag{3.15}$$
$$K(x, t) = \frac{\pi}{2} x^{1/2} t^{1/2} \int_0^\infty \lambda g(\lambda) J_{n+1/2}(\lambda t) J_{n+1/2}(\lambda x)\, d\lambda$$
$$g(\lambda) = -I(\mu_1) - k^{-1} J(\mu_1)$$

Therefore, the initial problems (1.3 and 1.4) are reduced, respectively, to the set of eigenvalue problems (3.15) and (3.14) for the shortening parameter $\lambda_1 < 1$ for each n-th harmonic $(n = 1, 2, \ldots)$. The kernels $K_{ij}(x, t)$ $(i, j = 1, 2)$ and $K(x, t)$ of the integral equations obtained are cotinuous everywhere except at the points satisfying the conditions $k(\lambda_1) q(\lambda_1) = 0$ and $k(\lambda_1) = 0$. The first determines the value $\lambda_1^* < 1$, corresponding to a surface instability of the half-space /20/, and the second to the value $\lambda_1^{*ax} \leqslant \lambda_1^*$, corresponding to the surface instability of the half-space on the basis of a consideration of just the axisymmetric linearized problem.

The critical values λ_1 obtained when investigating the eigenvalue problems according to the fracture criterion used correspond to the beginning of the fracture of a material weakened by a periodic system of parallel cracks under compression along the latter.

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ASYMPTOTIC SOLUTIONS OF INTEGRAL EQUATIONS OF CRACK THEORY PROBLEMS FOR THIN PLATES*

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Integral equations to which problems of the bending of thin plates with slits can be reduced are considered. On the basis of the properties of the integral equation kernels, conclusions are drawn concerning the classes of existence and uniqueness of their solutions. Asymptotic methods based on extraction of their principal part with subsequent exact inversion are proposed for the solution of the integral equations. On the basis of the solutions obtained, formulas are presented for the stress intensity factors in the slit angles, and their dependence on the geometrical parameters of the problem is shown. Other problems are indicated that result in the solution of the integral equations under consideration.

Asymptotic methods of solving integral problems of elasticity theory problems on cracks /1-3/ were considered earlier, as were also integral equations /4/ analogous to those considered below.

1. The integral equation. Two kinds of problems (A and B) of crack theory for Kirchhoff-Love plates are studied.

Problem A. A Kirchhoff-Love plate in the form of a strip of width $2h\ (0 \leqslant y \leqslant 2h)$ is considered which is stiffly clamped along the edges. There is a rectilinear slit (crack) of length $2a$ on the plate axis of symmetry $(y = h)$. The slit (crack) edges are subjected to the action of a bending moment $M_y = \varphi_1(x)$. It is required to determine the angle of rotation of the slit edge $g_1^0(x)$ (Fig.1a).

Problem B. As in problem A, a plate in the form of a strip with a slit (crack) is considered. The slit (crack) is opened under the action of an antisymmetric transverse force $V_y = \varphi_2(x)$ distributed along the slit edges. Determine the vertical displacement of the slit (crack) edges $g_2^0(x)$ (Fig.1b).

The mathematical formulation of the problems under consideration is as follows: find the solution of the boundary value problem for the biharmonic equation

$$D\Delta^2 w = q(x, y) \tag{1.1}$$

($w(x, y)$ is the plate deflection, $q(x, y)$ is the distributed load, and D is the cylindrical stiffness) with mixed boundary conditions.

Problem A.

$$w(x,0) = w_y'(x, 0) = V_y(x, h) = 0, \ |x| < \infty \tag{1.2}$$
$$M_y(x, h) = \varphi_1(x), \ |x| < a; \ w_y'(x, h) = 0, \ a < |x| < \infty$$

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